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Toward a Universal Lagrangian

Jacek M. Rayski, Jr.[1,2]

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A gauge theory with the gauge group $U(1)\times U(1)\times SU(2)\times SU(3)\times SU(4)$ is shown to fit well into the generalized Kaluza scheme with eleven-dimensional space-time and its compact subspace $S^2\times S^5$. A unified theory is obtained which exhibits some broken super-symmetric features ($N=8$). Our approach is dictated by phenomenological requirements. The appearance of three generations of leptons and six flavors of colored quarks follows naturally. Within our Lagrangian there appear several free parameters (coupling constants), but some relations between them may follow from the requirement of cancellation of divergencies.

1. TWO DIFFERENT STRATEGIES

In the last few years it became fashionable to combine the generalized multidimensional Kaluza-Klein (Kaluza, 1921; Klein, 1926) theory with supersymmetries. The reason is that if the space is compactified and reinterpreted from the viewpoint of an observer living in the Minkowskian subspace of the multidimensional space, even the simplest supermultiplet ($N_{(d)}=1$) involving one tensor and one spin-3/2 field in d dimensions represents a richer set of fields including also lower spin values. The number of dimensions 11 seems particularly interesting and promising for several reasons. However, using the formulas

$$n(2)=\tfrac{1}{2}(d-1)(d-2)-1 \qquad \text{and} \qquad n(3/2)=\tfrac{1}{2}2^{E(d/2)}(d-3) \tag{1}$$

for the numbers of field components in the case of massless fields with spins 2 and 3/2 in $d=11$ dimensions (whereby the number $n(3/2)$ applies to Weyl's spinors whereas in the case of Majorana spinors it should be doubled), it is seen that a doublet $\{2, 3/2\}$ does not involve the same number of bosonic and fermionic field components (44 components of spin 2 and

[1]Jagellonian University, Cracow, Poland.
[2]Temporarily at Dublin Institute for Advanced Studies, School of Theoretical Physics, 10 Burlington Rd., Dublin 4, Ireland.

128 of spin 3/2) so that 84 bosonic field components are missing. A necessary, though not sufficient, condition for the set of fields to form a supermultiplet is to supplement it by auxiliary bosonic fields whose number of components would be just 84. It has been noticed that 84 is the number of components of a completely antisymmetric tensor $A_{\lambda\mu\nu}$ in 11 dimensions, and been concluded that the set $N_{(11)}=1$ has to be supplemented by such an auxiliary field (Cremer and Julia, 1979; Binan et al., 1982; Duff, 1983).

It cannot be denied that this approach is satisfactory from a point of view of both superalgebra and differential geometry (the introduction of an antisymmetric tensor being connected with the concept of geometrical forms) but it has not much to do with physical reality, i.e., with symmetries believed to play a role in physics. The experience of the last few years contribute to the opinion that frontal attacks on the problem of unification based upon local supersymmetries are not very promising. Therefore we should rather look for some new, roundabout approaches, not so ambitious but methodologically safer.

One more argument against the above-mentioned strategy of starting with a supermultiplet in an open multidimensional space, introducing supersymmetric interactions and later, after compactification, trying to reinterpret the results from a four-dimensional viewpoint, is offered by a discussion of the case $d=5$. It follows from (1) that the number of metric tensor components in five dimensions is five, whereas that of spinor field components is only four. Thus, it cannot help to introduce auxiliary bosonic fields (as was the case in 11 dimensions) on the contrary, one would need one more spinor field component, but this does not make sense. A remedy could be to go over to the case of extended supermultiplet $N_{(5)}=2$ involving besides the metric field (five components) two spin-3/2 fields (eight components) and one five-vector field (three components). If reinterpreted from a four-dimensional viewpoint this set is seen to be reducible

$$\{2, 3/2, 3/2, 1\}_{(5)} \Rightarrow \{2, 3/2, 3/2, 1\}_{(4)} + \{1, 1/2, 1/2, 0, 0\}_{(4)} \tag{2}$$

An alternative but simpler possibility is to assume, together with Kaluza, that the metric field component g_{55} is not a field quantity but may be put equal to unity. As was shown by Einstein et al. (Bergmann, 1947) the condition $g_{55}=1$ may be secured by introducing a requirement that the world is closed in the fifth dimension and that every geodesic line closed in the fifth dimension closes without discontinuity of direction (in this case it may be called a closed fiber). Under this assumption the compactified five-dimensional space-time involves only four, not five, independent metric field components, reinterpretable from a four-dimensional viewpoint as a combination of the ordinary metric field $g_{\mu\nu}$ (two components) with a vector field A_ν (also two components) according to the following:

$$(\gamma_{\mathcal{M}\mathcal{N}}) = \left(\begin{array}{c|c} g_{\mu\nu} + e^2 l^2 A_\mu \mathcal{A}_\nu & el\mathcal{A}_\mu \\ \hline el\mathcal{A}_\nu & 1 \end{array}\right) \tag{3}$$

for $\mathcal{M}$, $\mathcal{N} = 0, 1, \ldots, 4$ and μ, $\nu = 0, \ldots, 3$. This assumption, equating the number of bosonic and fermionic field components, enables one to regard them as forming a supermultiplet $N_{(5)} = 1$, viz.,

$$\{2, 3/2\}_{(5)} \to \{2, 3/2\}_{(4)} + \{1, 1/2\}_{(4)} \tag{4}$$

which is also reducible, but simpler than (2).

The above considerations may be extended to the case of more than five dimensions (with $d-4$ compact dimensions) whereby the metric tensor splits according to

$$(\gamma_{\mathcal{M}\mathcal{N}}) = \left\{\begin{array}{c|c} g_{\mu\nu} + g^2 l^2 \sum\limits_a A^a_\mu A^a_\nu & gl \sum\limits_a K^a_m A^a_\mu \\ \hline gl \sum\limits_a K_m A_u & g_{mn} \end{array}\right\} \tag{5}$$

with m, $n = 4, \ldots, d-1$ and K^a_m denoting the Killing vectors of the compact subspace.

In close analogy to the five-dimensional case, we may disregard the formula (1), but instead assume that the sector g_{mn} is independent of the Minkowskian coordinates x^μ and consequently does not involve any scalar fields. This assumption is natural from the point of view of the fiber bundle technique. Thus, the number of independent field components appearing in equation (5) is equal to the number of vector fields plus two components of the Minkowskian $g_{\mu\nu}$. Consequently, the type and number of bosonic and fermionic field components in d dimensions will be different from that assumed usually.

2. A GAUGE THEORY

After the above preliminaries let us go over to a construction of a specific formalism. Instead of a frontal attack we shall use a step-by-step approach starting with a pure gauge theory, then generalize it in the sense of Kaluza, and, finally, look for possible connections with the idea of supersymmetries.

Admitting that guage interactions are the most common and the most important ones, we have to choose, above all, a suitable gauge group G. Obviously this group should contain the standard group $G = U(1) \times SU(2) \times SU(3)$ as its subgroup. The usual requirement that it should be a semisimple group may be regarded as too restrictive. Our choise is a

28-parameter group

$$G = U(1) \times SU(2) \times U(3) \times SU(4) \tag{6}$$

constituting a natural extension of the standard group and being a maximal subgroup of $SU(5) \times SU(5)$ which, in turn, is a maximal subgroup of the exceptional group E_8.

The vector representation of this group involves 28 vector fields to be denoted as follows: a singlet V_μ related with $U(1)$, a triplet A^a_γ related with $SU(2)$, a nonet V^a_γ connected with $U(3)$, and a 15-plet B^a_ν related with $SU(4)$. Obviously, the singlet and the triplet are to be identified with the Yang–Mills fields transmitting the electro-weak interactions, the nonet (reducible to an octet and a singlet) have to be responsible for the strong interactions and the 15-plet represents a new type of interaction constituting a generalization of the set of vector fields X_μ and Y_μ known from the GUT based on the group $SU(5)$.

The spinor fields constituting sources of all those vector fields may be assumed in the form of Majorana fields endowed with the usual spinor index $\alpha = 1, \ldots, 4$ connected with the Minkowski space M^4, and also with further indices connected with the representations of the subgroups of G. At first sight it seems that three such indices are needed, ρ, ω, and ξ for the groups $SU(2)$, $SU(3)$, and $SU(4)$, respectively, but in view of the fact that $U(3)$ is a subgroup of $SU(4)$ it is possible to assume a more economical scheme involving only two indices $\rho = 1, 2$ and $\xi = 1, \ldots, 4$. Thus, let us consider a set of fields, to be denoted as $\psi_{\alpha\rho\xi}$, and assume a Lagrangian

$$\mathscr{L} = \mathscr{L}^{(0)} + \mathscr{L}'_{(\text{el-weak})} + \mathscr{L}'_{(\text{strong})} + \mathscr{L}'_{(\text{new})} \tag{7}$$

where where $\mathscr{L}^{(0)}$ is a sum of Lagrangians for all free fields. The remaining terms denote interaction Lagrangians coupling with the field ψ with the gauge fields (and Higgs fields). In particular $\mathscr{L}'_{(\text{el-weak})}$ is a sum of Lagrangians describing the electro-weak interactions whereby summation is extended over the index $\xi = 1, \ldots, 4$. They involve the four-vector fields V_μ and A^a_μ ($a = 1, 2, 3$). Similarly $\mathscr{L}'_{(\text{new})}$ denotes an $SU(4)$-symmetric gauge interaction involving the index ξ and the set of 15 vector fields B^a_ν and a summation over the index ρ. It is seen that all spinor field components are engaged into these interactions so that they may be regarded as universal. On the other hand, the strong interactions are not universal and assume the form

$$\mathscr{L}'_{(\text{strong})} = \sum_{a=1}^{9} \sum_{\rho=1}^{2} \sum_{\xi,\eta=1}^{3} \Psi_{\rho\xi} T^a_{\xi\eta} V^a_\mu \gamma^\mu \psi_{\rho\eta} \tag{8}$$

where summation over the indices ξ, η is extended only over the set of three values ξ, $\eta = 1, 2, 3$ and $T^a_{\xi\eta}$ are 3×3 matrices of the group $U(3)$. It means that the numerals $\xi = 1, 2, 3$ denote three colors of quarks while $\xi = 4$ denotes leptons. In other words, the leptonic number is a fourth color although the generalized color symmetry is strongly broken by the interaction (8) involving the gluons V_ν^a. The bispinor $\psi_{\alpha\rho\xi}$ is interpretable as representing a doublet of quarks (e.g., the u and d quarks endowed with color) together with a doublet of leptons (e.g., e and ν_e).[3] The $SU(4)$-symmetric interactions involving the fields B_μ^a allow for a transformation of nucleons into leptons.

It may be concluded that a gauge theory based upon the gauge group (6) is a promising framework for a correct description of the fundamental physical objects and their dynamics, the more so that nothing prevents us from introducing further generations of quarks and leptons into the scheme and adjusting suitably the coupling constants of weak, strong, and the remaining $SU(4)$-symmetric interactions so as to achieve agreement with experimental evidence. On the other hand, this gauge theory has also some drawbacks, viz., it cannot predict the number of generations of leptons and flavors of quarks. The gauge group being not semisimple, no relations among the coupling constants are known. Moreover, the Higgs fields must be introduced "by hand." Last but not least, it does not necessitate the appearance of gravitation. Such drawbacks may be remedied, at least partly, by taking into account the idea of a multidimensional space-time, and later on of possible supersymmetries.

3. A MULTIDIMENSIONAL SPACE-TIME

An intimate connection between the gravitational field and the gauge fields is offered by the generalized Kaluza–Klein theory with the metric described in equation (5), the dimensionality and the symmetry of the compact subspace being dictated by the gauge group (6) (Rayski and Rayski, 1983).

Consider an 11-dimensional space-time with seven closed dimensions and with the topology of the following direct product of spaces: $M^4 \times S^2 \times S^5$, where M^4 means a four-dimensional Minkowski space and S^n is an n-dimensional surface of a sphere or hypersphere. The mixed tensor components $\gamma_{\nu n}$ with $\nu = 0, \ldots, 3$ and $n = 4, \ldots, 10$ are of the form

$$\gamma_{\nu n} = \sum_{a=1}^{3} A_\nu^a K_n^a \qquad \text{for } n = 4, 5$$

and (9)

$$\gamma_{\nu n} = \sum_{a=1}^{15} B_\nu^a K_n^a \qquad \text{for } n = 6, \ldots, 10$$

[3]Right-handed neutrinos still waiting to be discovered.

with the Killing vectors representing the symmetries of S^2 and S^5, respectively.

As is well known, the Lagrangian $R_{(11)}$ splits into the usual scalar curvature of the four-dimensional space-time $R_{(4)}$ and the Lagrangians for the fields A_ν^a and B_ν^a in the curved four-dimensional space-time with the metric $g_{\mu\nu}$ but otherwise free, i.e., belonging to $\mathscr{L}^{(0)}$ in (5). Obviously the Lagrangians for the free Majorana field and the interaction Lagrangians appearing in (5) have to be rewritten in a generally covariant way in terms of the metric $g_{\mu\nu}$. According to the formula for the number of components of a Majorana field in d dimensions

$$n(\tfrac{1}{2}) = 2^{E(d/2)} \tag{10}$$

it possesses 32 components, which is exactly equal to the number of components of the formerly introduced field $\psi_{\alpha\rho\xi}$. Thus the index α together with the suffices $\rho = 1, 2$ and $\xi = 1, \ldots, 4$ may be rearranged into a single index $\alpha = 1, \ldots, 32$ of a generalized spinor ψ_α in 11 dimensions.

In an 11-dimensional space (or direct product of spaces) there exist eleven 32×32 Dirac matrices $\Gamma_{\alpha\beta}^{\mathscr{M}}$ $(\alpha, \beta = 1, \ldots, 32)$ which may be also written in the form of direct products of two sets of usual 4×4 Dirac matrices $\gamma_{\alpha\beta}^k$ and $\beta_{\xi\eta}^k$ $(k = 1, \ldots, 4)$ and a set of 2×2 matrices $\tau_{\rho\sigma}^k$ $(k = 1, 2, 3)$

$$\begin{aligned} \Gamma^k &= \gamma^k \times \mathbb{1} \times \mathbb{1} && \text{for } k = 1, 2, 3 \\ \Gamma^{k+3} &= \gamma^4 \times \tau^k \times \mathbb{1} && \text{for } k = 1, 2 \\ \Gamma^{k+5} &= \gamma^5 \times \mathbb{1} \times \beta^k && \text{for } k = 1, \ldots, 5 \\ \Gamma^{11} &= \gamma^4 \times \tau^3 \times \mathbb{1} \end{aligned} \tag{11}$$

satisfying the usual anticommutation relations

$$\{\Gamma^{\mathscr{M}}, \Gamma^{\mathscr{N}}\} = 2\delta^{\mathscr{M}\mathscr{N}} \qquad \text{for } \mathscr{M}, \mathscr{N} = 1, \ldots, 11 \tag{12}$$

The matrix Γ^{11} may be assumed to be diagonal and related either to the timelike dimension and play the role of Γ^0 or, if Γ^0 is assumed to be a unit matrix, then Γ^{11} will play the same role as γ^5 does in Majorana theory. The matrices τ^1 and τ^2 are directly connected to S^2 and $\beta^1, \ldots, \beta^5$ to S^5. It is seen that weak interactions are related to S^2 whereby $U(1)$ is a maximal subgroup of $SU(2)$, whereas strong interactions are related to S^5 whereby $U(1) \times SU(3)$ is a maximal subgroup of $SU(4)$ [isomorphic with $SO(6)$, being a symmetry of S^5].

Coming back to the discussion of the Yang–Mills fields it is to be noticed that they are of two quite different provenances: the fields A_ν^a and B_ν^a are of metrical origin, i.e., are involved into the metric tensor components $\gamma_{\mu\nu}$ as well as into the mixed components $\gamma_{\mu m}$ (but not into γ_{mn}), whereas the singlet V_μ and the nonet V_ν^a are genuine vector components. But, being genuine vectors, they should not be four-vectors but 11-vectors $V_{\mathcal{M}}$, i.e., in the massless case they should possess nine independent components, two components in the subspace M^4 and seven more in the compact subspace $S^2 \times S^5$. Since the total number of such 11-vectors is ten, the number of their extra components $V_{\mathcal{M}}$ and $V_{\mathcal{M}}^a$ beyond the Minkowski space is 70. To the observer from the Minkowskian world these last appear as scalars, whence one should expect (besides the usual metric field $g_{\mu\nu}$) $3+15+1+9=28$ four-vector fields and 70 scalar fields. These scalars have not been introduced "by hand," but appear naturally, even compulsorily, within the 11-dimensional formalism.

In the usual gauge theories scalars are not gauge fields and it is not at all clear what type of interactions they should be submitted to. Now, inasmuch as, in fact, they appear not to be genuine scalars, but further components of multivectors, their interactions are nothing but gauge interactions in an 11-dimensional manifold. In particular, the strong interaction term (8) should be rewritten and completed as follows:

$$\mathscr{L}'_{(\text{strong})} = \sum_{a=1}^{g} \sum_{\rho,\sigma=1}^{2} \sum_{\xi,\eta,\zeta=1}^{3} \Psi_{\rho\xi} T^a_{\xi\zeta} V^a_{\mathcal{M}} \Gamma^{\mathcal{M}}_{\zeta\eta\rho\sigma} \psi_{\sigma\eta} \tag{8'}$$

where the usual spinor indices α, β have been omitted and the usual gravitation (in the sector $\mathcal{M}, \mathcal{N}=0, \ldots, 3$) has been neglected. A similar generalization applies to the singlet $V_{\mathcal{M}}$ involved in the weak interactions.

4. POSSIBLE RELATIONS TO SUPERSYMMETRIES

A remarkable result of the above considerations is that the set of gauge fields involved into $g_{\mathcal{M}\mathcal{N}}$, together with the nonet $V^a_{\mathcal{M}}$ and singlet $V_{\mathcal{M}}$, compatible with the world structure $M^4 \times S^2 \times S^5$ and with the gauge symmetry (4), is equivalent to a set of one tensor field and 28 vector fields together with 70 scalar fields if reinterpreted from the Minkowskian space-time. These numbers are exactly equal to those appearing in the well-known supermultiplet being the highest possible extension $N=8$, viz., Table I. This is a remarkable fact showing that our version of generalized Kaluza theory has something to do with supersymmetries. If this coincidence is not fallacious, it is not purely accidental—which is hard to believe—we should

Table I

Spin	2	3/2	1	1/2	0
Number of fields	1	8	28	56	70

expect that the (two-component complex or four-component real) fields describing massless particles with spins 3/2 and 1/2 whose numbers are 8 and 56, respectively, constitute the sources of our gauge fields and denote the fundamental building blocks of the unified theory.

To stress it once again: the 11-vectors (whose number is altogether ten, i.e., one plus one plus eight) are not introduced by hand. Their introduction has been dictated by the necessity of reconciliation of the topology and symmetry $S^2 \times S^5$ of the compact subspace of Kaluza's 11-dimensional theory with the requirements of global extended $N = 8$ supersymmetry.

According to (1) the number of components of Majorana field in 11 dimensions is 32. Nothing, however, forbids us from fusing some pairs of them into Dirac spinors in the ordinary space-time, though endowed with some additional indices (gauge group indices). Splitting the set of 56 spin-1/2 fields into $48+8$ the former may be reinterpreted as describing three generations of leptons, together with three generations (six flavors) of quarks $(48 \times 4 = (3+1) \times 4 \times 3 \times 4)$. The remaining 32 spin-1/2 components may be amalgamated with 32 spin-3/2 components to form vector-spinor fields, each of them describing particle and antiparticle states with four spin orientations $(3/2, 1/2, -1/2, -3/2)$. The effect of "swallowing" some spin-1/2 fields by spin-3/2 fields would be analogous to Higgs mechanism endowing the spin-3/2 fields with masses.

It should be also noticed that the total number of components of all spinor fields in Table I is equal to the number of components of massless spin-3/2 field in 11 dimensions. This cannot be a pure coincidence. Hence, it may be said that our set of fields consists of only one field of spin-2 and one field of spin-3/2 in 11 dimensions, supplemented by some auxiliary 11-vector fields (two singlets and one octet). The 11-vectors are indispensable to complete the $N = 8$ supermultiplet.

At this stage the formalism is locally gauge invariant but probably not supersymmetric as regards interactions between the members of the supermultiplet. Yet supersymmetry may well be realizable in Nature only partly, as strongly broken *su–sy*, with all interactions in the real world being only gauge interactions with coupling constants adjusted so as to maximally reduce divergencies.

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